

## 38. KINEMATICS

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Throughout this section units are used in which  $\hbar = c = 1$ . The following conversions are useful:  $\hbar c = 197.3 \text{ MeV fm}$ ,  $(\hbar c)^2 = 0.3894 \text{ (GeV)}^2 \text{ mb}$ .

### 38.1. Lorentz transformations

The energy  $E$  and 3-momentum  $\mathbf{p}$  of a particle of mass  $m$  form a 4-vector  $p = (E, \mathbf{p})$  whose square  $p^2 \equiv E^2 - |\mathbf{p}|^2 = m^2$ . The velocity of the particle is  $\boldsymbol{\beta} = \mathbf{p}/E$ . The energy and momentum  $(E^*, \mathbf{p}^*)$  viewed from a frame moving with velocity  $\boldsymbol{\beta}_f$  are given by

$$\begin{pmatrix} E^* \\ p_{\parallel}^* \end{pmatrix} = \begin{pmatrix} \gamma_f & -\gamma_f \beta_f \\ -\gamma_f \beta_f & \gamma_f \end{pmatrix} \begin{pmatrix} E \\ p_{\parallel} \end{pmatrix}, \quad p_T^* = p_T, \quad (38.1)$$

where  $\gamma_f = (1 - \beta_f^2)^{-1/2}$  and  $p_T$  ( $p_{\parallel}$ ) are the components of  $\mathbf{p}$  perpendicular (parallel) to  $\boldsymbol{\beta}_f$ . Other 4-vectors, such as the space-time coordinates of events, of course transform in the same way. The scalar product of two 4-momenta  $p_1 \cdot p_2 = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2$  is invariant (frame independent).

### 38.2. Center-of-mass energy and momentum

In the collision of two particles of masses  $m_1$  and  $m_2$  the total center-of-mass energy can be expressed in the Lorentz-invariant form

$$\begin{aligned} E_{\text{cm}} &= \left[ (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \right]^{1/2}, \\ &= \left[ m_1^2 + m_2^2 + 2E_1 E_2 (1 - \beta_1 \beta_2 \cos \theta) \right]^{1/2}, \end{aligned} \quad (38.2)$$

where  $\theta$  is the angle between the particles. In the frame where one particle (of mass  $m_2$ ) is at rest (lab frame),

$$E_{\text{cm}} = (m_1^2 + m_2^2 + 2E_{1\text{lab}} m_2)^{1/2}. \quad (38.3)$$

The velocity of the center-of-mass in the lab frame is

$$\boldsymbol{\beta}_{\text{cm}} = \mathbf{p}_{\text{lab}} / (E_{1\text{lab}} + m_2), \quad (38.4)$$

where  $\mathbf{p}_{\text{lab}} \equiv \mathbf{p}_{1\text{lab}}$  and

$$\gamma_{\text{cm}} = (E_{1\text{lab}} + m_2) / E_{\text{cm}}. \quad (38.5)$$

The c.m. momenta of particles 1 and 2 are of magnitude

$$p_{\text{cm}} = p_{\text{lab}} \frac{m_2}{E_{\text{cm}}}. \quad (38.6)$$

For example, if a 0.80 GeV/ $c$  kaon beam is incident on a proton target, the center of mass energy is 1.699 GeV and the center of mass momentum of either particle is 0.442 GeV/ $c$ . It is also useful to note that

$$E_{\text{cm}} dE_{\text{cm}} = m_2 dE_{1\text{lab}} = m_2 \beta_{1\text{lab}} dp_{\text{lab}}. \quad (38.7)$$

## 2 38. Kinematics

### 38.3. Lorentz-invariant amplitudes

The matrix elements for a scattering or decay process are written in terms of an invariant amplitude  $-i\mathcal{M}$ . As an example, the  $S$ -matrix for  $2 \rightarrow 2$  scattering is related to  $\mathcal{M}$  by

$$\begin{aligned} \langle p'_1 p'_2 | S | p_1 p_2 \rangle &= I - i(2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \\ &\times \frac{\mathcal{M}(p_1, p_2; p'_1, p'_2)}{(2E_1)^{1/2} (2E_2)^{1/2} (2E'_1)^{1/2} (2E'_2)^{1/2}} . \end{aligned} \quad (38.8)$$

The state normalization is such that

$$\langle p' | p \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') . \quad (38.9)$$

### 38.4. Particle decays

The partial decay rate of a particle of mass  $M$  into  $n$  bodies in its rest frame is given in terms of the Lorentz-invariant matrix element  $\mathcal{M}$  by

$$d\Gamma = \frac{(2\pi)^4}{2M} |\mathcal{M}|^2 d\Phi_n (P; p_1, \dots, p_n), \quad (38.10)$$

where  $d\Phi_n$  is an element of  $n$ -body phase space given by

$$d\Phi_n(P; p_1, \dots, p_n) = \delta^4(P - \sum_{i=1}^n p_i) \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3 2E_i} . \quad (38.11)$$

This phase space can be generated recursively, viz.

$$\begin{aligned} d\Phi_n(P; p_1, \dots, p_n) &= d\Phi_j(q; p_1, \dots, p_j) \\ &\times d\Phi_{n-j+1}(P; q, p_{j+1}, \dots, p_n) (2\pi)^3 dq^2 , \end{aligned} \quad (38.12)$$

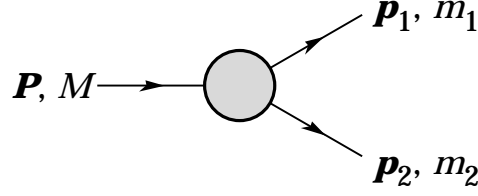
where  $q^2 = (\sum_{i=1}^j E_i)^2 - \left| \sum_{i=1}^j \mathbf{p}_i \right|^2$ . This form is particularly useful in the case where a particle decays into another particle that subsequently decays.

**38.4.1. Survival probability:** If a particle of mass  $M$  has mean proper lifetime  $\tau$  ( $= 1/\Gamma$ ) and has momentum  $(E, \mathbf{p})$ , then the probability that it lives for a time  $t_0$  or greater before decaying is given by

$$P(t_0) = e^{-t_0 \Gamma/\gamma} = e^{-Mt_0 \Gamma/E} , \quad (38.13)$$

and the probability that it travels a distance  $x_0$  or greater is

$$P(x_0) = e^{-Mx_0 \Gamma/|\mathbf{p}|} . \quad (38.14)$$



**Figure 38.1:** Definitions of variables for two-body decays.

### 38.4.2. Two-body decays :

In the rest frame of a particle of mass  $M$ , decaying into 2 particles labeled 1 and 2,

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M}, \quad (38.15)$$

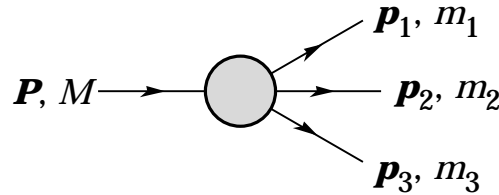
$$\begin{aligned} |\mathbf{p}_1| &= |\mathbf{p}_2| \\ &= \frac{[(M^2 - (m_1 + m_2)^2)(M^2 - (m_1 - m_2)^2)]^{1/2}}{2M}, \end{aligned} \quad (38.16)$$

and

$$d\Gamma = \frac{1}{32\pi^2} |\mathcal{M}|^2 \frac{|\mathbf{p}_1|}{M^2} d\Omega, \quad (38.17)$$

where  $d\Omega = d\phi_1 d(\cos\theta_1)$  is the solid angle of particle 1.

### 38.4.3. Three-body decays :



**Figure 38.2:** Definitions of variables for three-body decays.

Defining  $p_{ij} = p_i + p_j$  and  $m_{ij}^2 = p_{ij}^2$ , then  $m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$  and  $m_{12}^2 = (P - p_3)^2 = M^2 + m_3^2 - 2ME_3$ , where  $E_3$  is the energy of particle 3 in the rest frame of  $M$ . In that frame, the momenta of the three decay particles lie in a plane. The relative orientation of these three momenta is fixed if their energies are known. The momenta can therefore be specified in space by giving three Euler angles  $(\alpha, \beta, \gamma)$  that specify the orientation of the final system relative to the initial particle [1]. Then

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M} |\mathcal{M}|^2 dE_1 dE_2 d\alpha d(\cos\beta) d\gamma. \quad (38.18)$$

## 4 38. Kinematics

Alternatively

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M^2} |\mathcal{M}|^2 |\mathbf{p}_1^*| |\mathbf{p}_3| dm_{12} d\Omega_1^* d\Omega_3, \quad (38.19)$$

where  $(|\mathbf{p}_1^*|, \Omega_1^*)$  is the momentum of particle 1 in the rest frame of 1 and 2, and  $\Omega_3$  is the angle of particle 3 in the rest frame of the decaying particle.  $|\mathbf{p}_1^*|$  and  $|\mathbf{p}_3|$  are given by

$$|\mathbf{p}_1^*| = \frac{[(m_{12}^2 - (m_1 + m_2)^2)(m_{12}^2 - (m_1 - m_2)^2)]^{1/2}}{2m_{12}}, \quad (38.20a)$$

and

$$|\mathbf{p}_3| = \frac{[(M^2 - (m_{12} + m_3)^2)(M^2 - (m_{12} - m_3)^2)]^{1/2}}{2M}. \quad (38.20b)$$

[Compare with Eq. (38.16).]

If the decaying particle is a scalar or we average over its spin states, then integration over the angles in Eq. (38.18) gives

$$\begin{aligned} d\Gamma &= \frac{1}{(2\pi)^3} \frac{1}{8M} \overline{|\mathcal{M}|^2} dE_1 dE_2 \\ &= \frac{1}{(2\pi)^3} \frac{1}{32M^3} \overline{|\mathcal{M}|^2} dm_{12}^2 dm_{23}^2. \end{aligned} \quad (38.21)$$

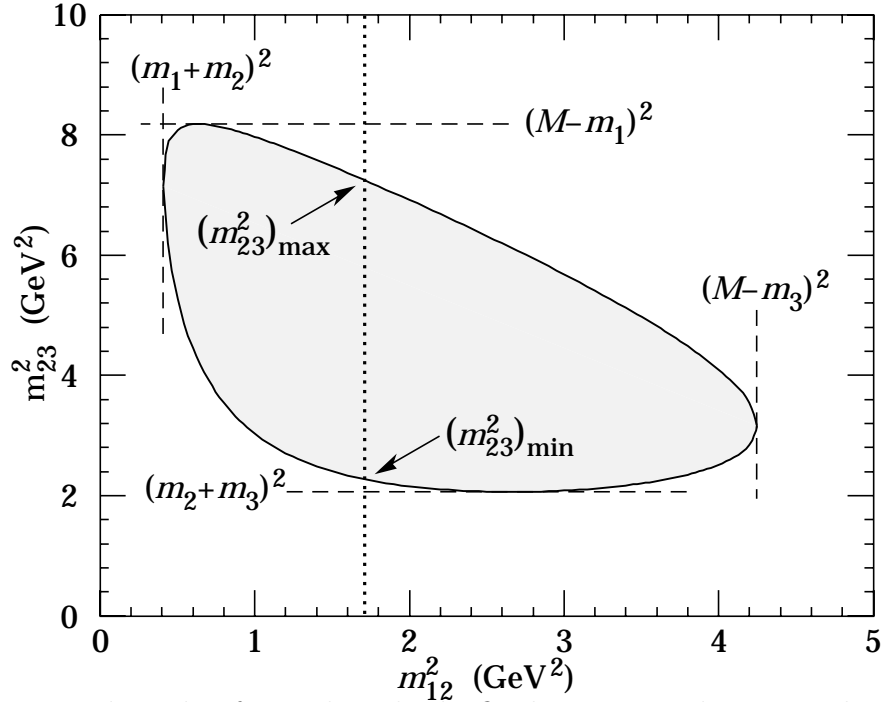
This is the standard form for the Dalitz plot.

**38.4.3.1. Dalitz plot:** For a given value of  $m_{12}^2$ , the range of  $m_{23}^2$  is determined by its values when  $\mathbf{p}_2$  is parallel or antiparallel to  $\mathbf{p}_3$ :

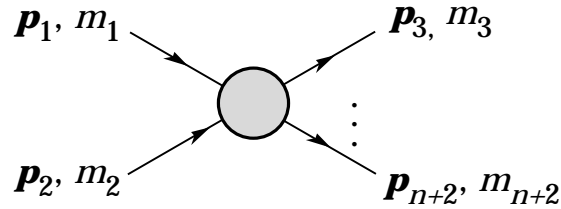
$$\begin{aligned} (m_{23}^2)_{\max} &= \\ &= (E_2^* + E_3^*)^2 - \left( \sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2} \right)^2, \end{aligned} \quad (38.22a)$$

$$\begin{aligned} (m_{23}^2)_{\min} &= \\ &= (E_2^* + E_3^*)^2 - \left( \sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2} \right)^2. \end{aligned} \quad (38.22b)$$

Here  $E_2^* = (m_{12}^2 - m_1^2 + m_2^2)/2m_{12}$  and  $E_3^* = (M^2 - m_{12}^2 - m_3^2)/2m_{12}$  are the energies of particles 2 and 3 in the  $m_{12}$  rest frame. The scatter plot in  $m_{12}^2$  and  $m_{23}^2$  is called a Dalitz plot. If  $\overline{|\mathcal{M}|^2}$  is constant, the allowed region of the plot will be uniformly populated with events [see Eq. (38.21)]. A nonuniformity in the plot gives immediate information on  $|\mathcal{M}|^2$ . For example, in the case of  $D \rightarrow K\pi\pi$ , bands appear when  $m_{(K\pi)} = m_{K^*(892)}$ , reflecting the appearance of the decay chain  $D \rightarrow K^*(892)\pi \rightarrow K\pi\pi$ .



**Figure 38.3:** Dalitz plot for a three-body final state. In this example, the state is  $\pi^+\bar{K}^0p$  at 3 GeV. Four-momentum conservation restricts events to the shaded region.



**Figure 38.4:** Definitions of variables for production of an  $n$ -body final state.

**38.4.4. Kinematic limits :** In a three-body decay the maximum of  $|\mathbf{p}_3|$ , [given by Eq. (38.20)], is achieved when  $m_{12} = m_1 + m_2$ , *i.e.*, particles 1 and 2 have the same vector velocity in the rest frame of the decaying particle. If, in addition,  $m_3 > m_1, m_2$ , then  $|\mathbf{p}_3|_{\max} > |\mathbf{p}_1|_{\max}, |\mathbf{p}_2|_{\max}$ .

**38.4.5. Multibody decays :** The above results may be generalized to final states containing any number of particles by combining some of the particles into “effective particles” and treating the final states as 2 or 3 “effective particle” states. Thus, if  $p_{ijk\dots} = p_i + p_j + p_k + \dots$ , then

$$m_{ijk\dots} = \sqrt{p_{ijk\dots}^2}, \quad (38.23)$$

and  $m_{ijk\dots}$  may be used in place of *e.g.*,  $m_{12}$  in the relations in Sec. 38.4.3 or 38.4.3.1 above.

## 6 38. Kinematics

### 38.5. Cross sections

The differential cross section is given by

$$d\sigma = \frac{(2\pi)^4 |\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \times d\Phi_n(p_1 + p_2; p_3, \dots, p_{n+2}) . \quad (38.24)$$

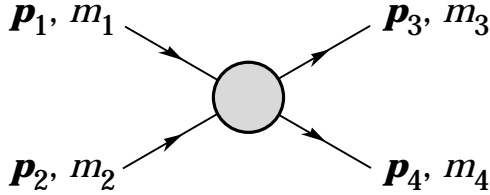
[See Eq. (38.11).] In the rest frame of  $m_2$ (lab),

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = m_2 p_{1 \text{ lab}} ; \quad (38.25a)$$

while in the center-of-mass frame

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = p_{1 \text{ cm}} \sqrt{s} . \quad (38.25b)$$

#### 38.5.1. Two-body reactions :



**Figure 38.5:** Definitions of variables for a two-body final state.

Two particles of momenta  $p_1$  and  $p_2$  and masses  $m_1$  and  $m_2$  scatter to particles of momenta  $p_3$  and  $p_4$  and masses  $m_3$  and  $m_4$ ; the Lorentz-invariant Mandelstam variables are defined by

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = m_1^2 + 2E_1 E_2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 + m_2^2 , \quad (38.26)$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = m_1^2 - 2E_1 E_3 + 2\mathbf{p}_1 \cdot \mathbf{p}_3 + m_3^2 , \quad (38.27)$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_1^2 - 2E_1 E_4 + 2\mathbf{p}_1 \cdot \mathbf{p}_4 + m_4^2 , \quad (38.28)$$

and they satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 . \quad (38.29)$$

The two-body cross section may be written as

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s} \frac{1}{|\mathbf{p}_{1\text{cm}}|^2} |\mathcal{M}|^2 . \quad (38.30)$$

In the center-of-mass frame

$$\begin{aligned} t &= (E_{1\text{cm}} - E_{3\text{cm}})^2 - (p_{1\text{cm}} - p_{3\text{cm}})^2 - 4p_{1\text{cm}} p_{3\text{cm}} \sin^2(\theta_{\text{cm}}/2) \\ &= t_0 - 4p_{1\text{cm}} p_{3\text{cm}} \sin^2(\theta_{\text{cm}}/2) , \end{aligned} \quad (38.31)$$

where  $\theta_{\text{cm}}$  is the angle between particle 1 and 3. The limiting values  $t_0$  ( $\theta_{\text{cm}} = 0$ ) and  $t_1$  ( $\theta_{\text{cm}} = \pi$ ) for  $2 \rightarrow 2$  scattering are

$$t_0(t_1) = \left[ \frac{m_1^2 - m_3^2 - m_2^2 + m_4^2}{2\sqrt{s}} \right]^2 - (p_{1\text{cm}} \mp p_{3\text{cm}})^2 . \quad (38.32)$$

In the literature the notation  $t_{\text{min}}$  ( $t_{\text{max}}$ ) for  $t_0$  ( $t_1$ ) is sometimes used, which should be discouraged since  $t_0 > t_1$ . The center-of-mass energies and momenta of the incoming particles are

$$E_{1\text{cm}} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} , \quad E_{2\text{cm}} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} , \quad (38.33)$$

For  $E_{3\text{cm}}$  and  $E_{4\text{cm}}$ , change  $m_1$  to  $m_3$  and  $m_2$  to  $m_4$ . Then

$$p_{i\text{cm}} = \sqrt{E_{i\text{cm}}^2 - m_i^2} \text{ and } p_{1\text{cm}} = \frac{p_{1\text{lab}} m_2}{\sqrt{s}} . \quad (38.34)$$

Here the subscript lab refers to the frame where particle 2 is at rest. [For other relations see Eqs. (38.2)–(38.4).]

**38.5.2. Inclusive reactions :** Choose some direction (usually the beam direction) for the  $z$ -axis; then the energy and momentum of a particle can be written as

$$E = m_T \cosh y , \quad p_x , p_y , p_z = m_T \sinh y , \quad (38.35)$$

where  $m_T$  is the transverse mass

$$m_T^2 = m^2 + p_x^2 + p_y^2 , \quad (38.36)$$

and the rapidity  $y$  is defined by

$$\begin{aligned} y &= \frac{1}{2} \ln \left( \frac{E + p_z}{E - p_z} \right) \\ &= \ln \left( \frac{E + p_z}{m_T} \right) = \tanh^{-1} \left( \frac{p_z}{E} \right) . \end{aligned} \quad (38.37)$$

## 8 38. Kinematics

Under a boost in the  $z$ -direction to a frame with velocity  $\beta$ ,  $y \rightarrow y - \tanh^{-1} \beta$ . Hence the shape of the rapidity distribution  $dN/dy$  is invariant. The invariant cross section may also be rewritten

$$E \frac{d^3\sigma}{d^3p} = \frac{d^3\sigma}{d\phi dy p_T dp_T} \implies \frac{d^2\sigma}{\pi dy d(p_T^2)}. \quad (38.38)$$

The second form is obtained using the identity  $dy/dp_z = 1/E$ , and the third form represents the average over  $\phi$ .

Feynman's  $x$  variable is given by

$$x = \frac{p_z}{p_{z \max}} \approx \frac{E + p_z}{(E + p_z)_{\max}} \quad (p_T \ll |p_z|). \quad (38.39)$$

In the c.m. frame,

$$x \approx \frac{2p_{z \text{ cm}}}{\sqrt{s}} = \frac{2m_T \sinh y_{\text{cm}}}{\sqrt{s}} \quad (38.40)$$

and

$$= (y_{\text{cm}})_{\max} = \ln(\sqrt{s}/m). \quad (38.41)$$

For  $p \gg m$ , the rapidity [Eq. (38.37)] may be expanded to obtain

$$\begin{aligned} y &= \frac{1}{2} \ln \frac{\cos^2(\theta/2) + m^2/4p^2 + \dots}{\sin^2(\theta/2) + m^2/4p^2 + \dots} \\ &\approx -\ln \tan(\theta/2) \equiv \eta \end{aligned} \quad (38.42)$$

where  $\cos \theta = p_z/p$ . The pseudorapidity  $\eta$  defined by the second line is approximately equal to the rapidity  $y$  for  $p \gg m$  and  $\theta \gg 1/\gamma$ , and in any case can be measured when the mass and momentum of the particle is unknown. From the definition one can obtain the identities

$$\sinh \eta = \cot \theta, \quad \cosh \eta = 1/\sin \theta, \quad \tanh \eta = \cos \theta. \quad (38.43)$$

**38.5.3. Partial waves:** The amplitude in the center of mass for elastic scattering of spinless particles may be expanded in Legendre polynomials

$$f(k, \theta) = \frac{1}{k} \sum_{\ell} (2\ell + 1) a_{\ell} P_{\ell}(\cos \theta), \quad (38.44)$$

where  $k$  is the c.m. momentum,  $\theta$  is the c.m. scattering angle,  $a_{\ell} = (\eta_{\ell} e^{2i\delta_{\ell}} - 1)/2i$ ,  $0 \leq \eta_{\ell} \leq 1$ , and  $\delta_{\ell}$  is the phase shift of the  $\ell^{\text{th}}$  partial wave. For purely elastic scattering,  $\eta_{\ell} = 1$ . The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(k, \theta)|^2. \quad (38.45)$$



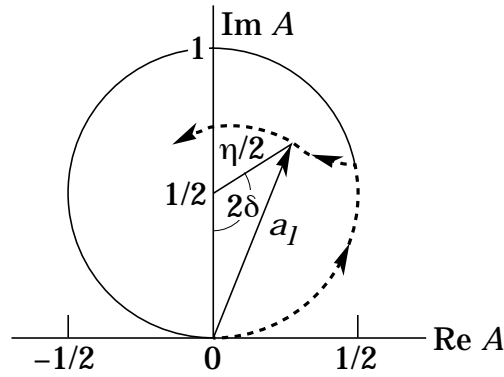
The optical theorem states that

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \text{Im} f(k, 0) , \quad (38.46)$$

and the cross section in the  $\ell^{\text{th}}$  partial wave is therefore bounded:

$$\sigma_{\ell} = \frac{4\pi}{k^2} (2\ell + 1) |a_{\ell}|^2 \leq \frac{4\pi(2\ell + 1)}{k^2} . \quad (38.47)$$

The evolution with energy of a partial-wave amplitude  $a_{\ell}$  can be displayed as a trajectory in an Argand plot, as shown in Fig. 38.6.



**Figure 38.6:** Argand plot showing a partial-wave amplitude  $a_{\ell}$  as a function of energy. The amplitude leaves the unitary circle where inelasticity sets in ( $\eta_{\ell} < 1$ ).

The usual Lorentz-invariant matrix element  $\mathcal{M}$  (see Sec. 38.3 above) for the elastic process is related to  $f(k, \theta)$  by

$$\mathcal{M} = -8\pi\sqrt{s} f(k, \theta) , \quad (38.48)$$

so

$$\sigma_{\text{tot}} = -\frac{1}{2p_{\text{lab}} m_2} \text{Im} \mathcal{M}(t = 0) , \quad (38.49)$$

where  $s$  and  $t$  are the center-of-mass energy squared and momentum transfer squared, respectively (see Sec. 38.4.1).

## 10 38. Kinematics

**38.5.3.1. Resonances:** The Breit-Wigner (nonrelativistic) form for an elastic amplitude  $a_\ell$  with a resonance at c.m. energy  $E_R$ , elastic width  $\Gamma_{\text{el}}$ , and total width  $\Gamma_{\text{tot}}$  is

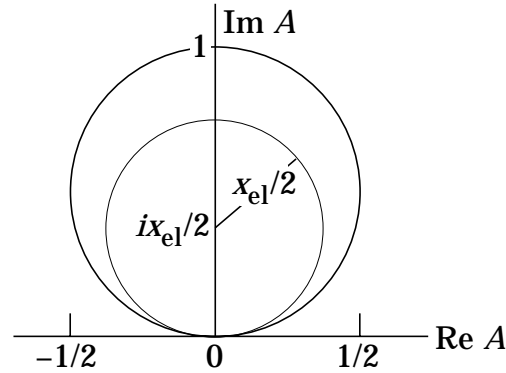
$$a_\ell = \frac{\Gamma_{\text{el}}/2}{E_R - E - i\Gamma_{\text{tot}}/2}, \quad (38.50)$$

where  $E$  is the c.m. energy. As shown in Fig. 38.7, in the absence of background the elastic amplitude traces a counterclockwise circle with center  $ix_{\text{el}}/2$  and radius  $x_{\text{el}}/2$ , where the elasticity  $x_{\text{el}} = \Gamma_{\text{el}}/\Gamma_{\text{tot}}$ . The amplitude has a pole at  $E = E_R - i\Gamma_{\text{tot}}/2$ .

The spin-averaged Breit-Wigner cross section for a spin- $J$  resonance produced in the collision of particles of spin  $S_1$  and  $S_2$  is

$$\sigma_{BW}(E) = \frac{(2J+1)}{(2S_1+1)(2S_2+1)} \frac{\pi}{k^2} \frac{B_{\text{in}}B_{\text{out}}\Gamma_{\text{tot}}^2}{(E - E_R)^2 + \Gamma_{\text{tot}}^2/4}, \quad (38.51)$$

where  $k$  is the c.m. momentum,  $E$  is the c.m. energy, and  $B_{\text{in}}$  and  $B_{\text{out}}$  are the branching fractions of the resonance into the entrance and exit channels. The  $2S+1$  factors are the multiplicities of the incident spin states, and are replaced by 2 for photons. This expression is valid only for an isolated state. If the width is not small,  $\Gamma_{\text{tot}}$  cannot be treated as a constant independent of  $E$ . There are many other forms for  $\sigma_{BW}$ , all of which are equivalent to the one given here in the narrow-width case. Some of these forms may be more appropriate if the resonance is broad.



**Figure 38.7:** Argand plot for a resonance.

The relativistic Breit-Wigner form corresponding to Eq. (38.50) is:

$$a_\ell = \frac{-m\Gamma_{\text{el}}}{s - m^2 + im\Gamma_{\text{tot}}}. \quad (38.52)$$

A better form incorporates the known kinematic dependences, replacing  $m\Gamma_{\text{tot}}$  by  $\sqrt{s}\Gamma_{\text{tot}}(s)$ , where  $\Gamma_{\text{tot}}(s)$  is the width the resonance particle would have if its mass

were  $\sqrt{s}$ , and correspondingly  $m\Gamma_{\text{el}}$  by  $\sqrt{s}\Gamma_{\text{el}}(s)$  where  $\Gamma_{\text{el}}(s)$  is the partial width in the incident channel for a mass  $\sqrt{s}$ :

$$a_\ell = \frac{-\sqrt{s}\Gamma_{\text{el}}(s)}{s - m^2 + i\sqrt{s}\Gamma_{\text{tot}}(s)}. \quad (38.53)$$

For the  $Z$  boson, all the decays are to particles whose masses are small enough to be ignored, so on dimensional grounds  $\Gamma_{\text{tot}}(s) = \sqrt{s}\Gamma_0/m_Z$ , where  $\Gamma_0$  defines the width of the  $Z$ , and  $\Gamma_{\text{el}}(s)/\Gamma_{\text{tot}}(s)$  is constant. A full treatment of the line shape requires consideration of dynamics, not just kinematics. For the  $Z$  this is done by calculating the radiative corrections in the Standard Model.

**References:**

1. See, for example, J.J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley (1985), p. 172, or D.M. Brink and G.R. Satchler, *Angular Momentum*, 2nd ed., Oxford University Press (1968), p. 20.